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HEREDITARY DEPENDENCE IN THE THEORY

OF DIFFERENTIAL EQUATIONS*-PART I

by

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1. <u>Introduction</u>

The notion of a differential equation with hereditary dependence is by no means new, and, in fact, dates back to the very early days of the notion of a differential equation itself. Apparently the first mathematician to seriously consider and investigate such dependence was John Bernoulli who published his results in 1728. Early considerations are also to be bound in the work of Euler. However, the problems in mechanics which had initially suggested the introduction of hereditary differential equations were later found to be more conveniently handled in terms of partial differential equations. Hence, as a consequence of a lack of further motivation at the time and, of course, the need to first develop a substantial theory for differential equations in the ordinary sense, further development of hereditary dependence was left for the twentieth century. The next milestone was the brilliant and farsighted accomplishments of Volterrra in his investigation of the growth properties of interacting species of organisms. With the exclusion of this work, the present theory is substantially the product of the past twenty years. Its development follows as a natural response to the many problems recently identified in controls analysis, econometrics, chemistry, biology, and medicine which are most properly described in terms of differential equations with hereditary dependence.

As might be expected the first hereditary differential equations to receive general treatment were the so-called differential-difference equations with constant lags. A substantial portion of the early work on these equations was carried out by E. M. Wright and A. D. Myškis in the late 1940's. During the 1950's substantial progress was made through the efforts of a large group of Russian mathematicians and in this country and Europe by R. E. Bellman, K. L. Cooke, W. Hahn, and others. In 1963 a comprehensive book by R. E. Bellman and K. L. Cooke became available on this special class of hereditary equations. Major contributions in a somewhat more general setting have been made by N. N. Krasovskiĭ, A. Halanay, J. K. Hale, L. E. El'sgol'ts, S. M. Shimanov, and other Russian and American mathematicians. A general, though far from complete, list of references is given at the end of this paper.

In this paper the author introduces a general class of differential equations with hereditary dependence which includes most equations of hereditary type encountered in applications with the notable exception of equations of neutral type. Furthermore, the class introduced allows the development of a strong and tractable qualitative theory closely resembling that of ordinary differential equations. For example, much of the linear analysis and the stability theory as developed by J. K. Hale in [21] and [22] for a more restricted class of hereditary equations carries over directly. However, only results associated with existence and uniqueness of solutions and dependence on initial data and parameters will be considered herein.

The contents of this paper is an expansion and somewhat reformulated version of material developed by the author and J. K. Hale for seminar presentation at RIAS two years ago. More recently it has constituted part of course presented by the author at the University of Maryland. The author is indebted to the mathematics staff at RIAS, and in particular to Dr. J. K. Hale, for many suggestions which substantially improved the quality of this work.

To get our discussion under way let us exhibit a few important special differential forms which are included in our general class. First of all, of course, ordinary differential equations

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

are included. Also differential-difference equations, say of the form

$$\dot{x}(t) = f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_n)), \tag{1.2}$$

$\tau_n > \tau_{n-1} > \ldots > \tau_1 > 0$, are included as are equations involving an infinite set of discrete retarded arguments such as, for example,

$$\dot{x}(t) = -\alpha \sum_{n=1}^{\infty} \frac{x(t - n)}{2^{n-1}} (1 + x(t)). \tag{1.3}$$

Equations where change of state involves dependence on the state over an interval of fixed length,

$$\dot{x}(t) = f(t, x \mid [t - 1, t]), \tag{1.4}$$

are included, as are equations with dependence on the entire state history,

$$\dot{x}(t) = f(t, x \mid (-\infty, t]). \tag{1.5}$$

Other types included are illustrated by the forms

$$\dot{x}(t) = f(t, x(t), x(\tfrac{t}{2})), \tag{1.6}$$

$$\dot{x}(t) = f(t, x \mid [t_0, t]), \tag{1.7}$$

and

$$\dot{x}(t) = f(t, x \mid [\tfrac{t - 1}{t + 1} , t]). \tag{1.8}$$

Many more different types are contained in our general class, but enough have been presented to be suggested of the broad scope of this study. We remark in passing that equation (1.5) in a sense includes all the other forms listed as special cases. However, in as much as it is often true that stronger results are possible when the domains of hereditary dependence is more closely identified, it seems proper to adopt notation which allows us to accomplish this easily.

2. General Concept of Hereditary Dependence

Let us now proceed with the identification of our general class of hereditary differential equations. In the way of general notation we specify the real line by R, the space of n-dimensional real vectors by R^n, and let $| \ |$ denote any convenient norm on R^n. Let Ω denote the set of all closed subsets of R bounded from above.

A function $\alpha: R \to \Omega$ is specified to be a lag function if $t \in \alpha(t)$ and $\alpha(t) \subset (-\infty, t]$ for all t in R. For lag functions α we shall consistently denote function values $\alpha(t)$ by α_t.

For arbitrary $\omega \in \Omega$ and an open and connected set $B \subset R^n$, let $Q(\omega, B)$ denote some distinguished class of functions mapping ω into B. For example, Q in a given situation might represent continuous functions, piecewise continuous functions, measurable functions, or some other convenient class. If we wish to specifically denote continuous or measurable functions, then Q will be replaced by C or M respectively. We note in passing that no restriction is imposed which excludes the possibility of $Q(\omega, B)$ being a single function.

In our discussion it is convenient to specify a function as right-continuous if it is piecewise continuous and continuous from the right.

For a given lag function α and a function x in $Q(R, B)$, we adopt the notation $x(\alpha_t)$ for the mapping obtained by restricting x to the set α_t. On other occasions, however, when considering x restricted to a set ω in R, we shall use the standard notation $x|\omega$.

It is also convenient to introduce the notation $U_t(B)$ for the set of all functions $\phi: R \to R^n$ such that $\phi|(-\infty, t]$ is contained in some specified $Q((-\infty, t], R^n)$, $\phi|[t, \infty)$ is contained in $C([t, \infty), B)$, and it is understood that $U_t(B)$ is invariant under translations to the left. That is, x in $U_t(B)$, $\tau > 0$, and y such that $y(t) = x(t + \tau)$ for t in R imply that y is contained in $U_t(B)$.

Consider now that we have a function G defined for t in some interval $[t_o, t_o + \tau_o]$, $\tau_o > 0$, and x in $U_{t_o}(B)$. We formulate a hereditary differential equation of general type to be a functional relationship of the form

$$\dot{x}(t) = G(t, x(\alpha_t)), \qquad t \geq t_o \tag{2.1}$$

That is, we are dealing with systems where the change in state at a specified time t is functionally dependent on t and the state history of the system taken with respect to the time set α_t. In equation (2.1) $\dot{x}(t)$ will in general denote the right hand derivative of x at t. Considering a lag function α, we denote by $\tilde{\alpha}_t$ the smallest interval containing α_t. The set $\overleftarrow{\alpha}_t$ defined by the formula

$$\overleftarrow{\alpha}_t = \bigcup \{ \tilde{\alpha}_\tau \cap (-\infty, t] : \ \tau \text{ in } [t, \infty) \} \tag{2.2}$$

is called the <u>domain of initial specification</u> at t for reasons that will be apparent. In setting up an initial data problem for equation (2.1) one specifies an n-vector function ϕ on R. A function x with $x(\overleftarrow{\alpha}_{t_o}) = \phi(\overleftarrow{\alpha}_{t_o})$ and satisfying equation (2.1) for all t in some interval $[t_o, t_o + \tau)$, $\tau > 0$, is referred to as a solution of equation (2.1). The function ϕ or more specifically its restriction $\phi(\overleftarrow{\alpha}_{t_o})$, is referred to as an initial function.

For ω in Ω, we specify $|| \ ||$ to be a norm defined on spaces $Q(\omega, R^n)$. $|| \ ||$ in turn is used to introduce a notion of continuity into the functional relationship defined by equation (2.1). For our study it is appropriate that we require $|| \ ||$ to have the following special properties:

(1) If $t = \max\{\tau : \tau \text{ in } \omega\}$ is contained in $u \subset \omega$, and x is a function in $Q(\omega, R^n)$, then $||x|u|| \leq ||x||$.

(2) If $\omega = u \cup v$ and x is a function in $Q(\omega, R^n)$ then $||x|| \leq ||x|u|| + ||x|v||$.

(3) For all $\epsilon \geq 0$ there exists $b > 0$ such that if x in $Q(\omega, R^n)$ is such that $|x(\theta)| \leq \epsilon$ for all θ in ω, then $||x|| < b\epsilon$.

(4) If x is continuous on ω and $||x|| = 0$, then x is identically zero.

(5) For all $\epsilon \geq 0$ there exists $a > 0$ such that if x in $Q(\omega, R^n)$ is such that $||x|| \leq \epsilon$, then $|x(t)| \leq a\epsilon$ where $t = \max\{\tau : \tau \text{ in } \omega\}$.

For x in $C(\omega, B)$, B bounded, one can easily observe that the norm defined by the formula

$$||x|| = \sup\{|x(\theta)| : \theta \text{ in } \omega\},$$

satisfies the imposed conditions. On the other hand, if we assume $Q(\omega, R^n) = M(\omega, R^n)$, it is clear that conditions (3) and (5) are not in general satisfied by the formula

$$||x|| = \left(\int_\omega |x(\theta)|^2 d\theta\right)^{\frac{1}{2}}.$$

When ω is compact, a convenient norm which does meet our conditions is given by the formula

$$||x|| = \left(\int_\omega |x(\theta)|^2 d\theta + |x(t)|^2\right)^{\frac{1}{2}}$$

where $t = \max\{\tau : \tau \text{ in } \omega\}$.

Our first existence theorem for the initial data problem we have formulated with respect to equation (2.1) bears a strong resemblance to the well known Chauchy-Peano Theorem for ordinary differential equations.

Theorem 2.1: Suppose for ψ in $U_{t_0}(B)$, $G(t, \psi(\alpha_t))$ is a continuous function of $\psi(\alpha_t)$ uniformly for t in some interval $[t_0, t_0 + \tau_0)$ and right-continuous in t on $[t_0, t_0 + \tau_0)$ for each ψ in $U_{t_0}(B)$. Then for each ϕ in $U_{t_0}(B)$ there exists $\tau_1 > 0$ such that equation (2.1) has a solution x defined on $[t_0, t_0 + \tau_1)$ corresponding to ϕ at t_0.

Proof. Specify ϕ in $U_{t_0}(B)$ and let $\tilde{\phi}$ be a function defined on R by the formula

$$\tilde{\phi}\,|(-\infty, t_0] = \phi\,|\,(-\infty, t_0]$$

$$\tilde{\phi}(t) = \phi(t_0), \text{ for } t \geq t_0.$$

Our continuity conditions imply that $|G(t, \tilde{\phi}(\alpha_t))|$ has a limit at $t \to t_0^+$ and we define

$$M_0 = \lim_{t \to t_0^+} |G(t, \tilde{\phi}(\alpha_t))|.$$

For arbitrary positive constants b and τ, let $S(b,\tau)$ be the set of all functions ψ defined on $(-\infty, t_0 + \tau]$, continuous on $[t_0, t_0 + \tau]$ and such that

$$\psi|(-\infty, t_0] = \phi\,|\,(-\infty, t_0]$$

$$|\psi(t) - \phi(t_0)| \leq b$$

and

$$|\psi(t_1) - \psi(t_2)| \leq (M_0 + 1)\,|t_1 - t_2|$$

for all t, t_1 and t_2 in $[t_0, t_0 + \tau]$. Clearly the sets $S(b,\tau)$ are convex and the Arzelà- Ascoli Theorem implies they are compact in the norm topology defined by the functional

$$\|x\| = \sup \{ |\psi(t)| : t \text{ in } [t_0, t_0 + \tau] \}$$

Choose $b_1 > 0$ such that $\{y : | y - \phi(t_0) | \leqq b_1\}$ is contained in B.

Letting ψ be an arbitrary function in $S(b_1, \tau)$ we have

$$| \psi(\alpha_t)(\theta) - \tilde{\phi}(\alpha_t)(\theta) | = 0 \tag{2.3}$$

if $\theta \leqq t_0$. If $t_0 < \theta \leqq t$, $t \leqq t_0 + \tau$, we have

$$| \psi(\alpha_t)(\theta) - \tilde{\phi}(\alpha_t)(\theta) | = | \psi(\alpha_t)(\theta) - \phi(t_0) | \leqq (M_0 + 1) |\theta - t_0| . \tag{2.4}$$

Now by the continuity of $G(t, \psi(\alpha_t))$ in $\psi(\alpha_t)$ for ψ in $U_{t_0}(B)$ and fixed t, we have the existence of a number $\delta > 0$ such that

$\| \psi(\alpha_t) - \tilde{\phi}(\alpha_t) \| < \delta$ implies

$$|G(t, \psi(\alpha_t)) - G(t, \tilde{\varphi}(\alpha_t))| < \frac{1}{2} ,$$

for arbitrary t in $[t_o, t_o + \tau)$. On the other hand property (3) imposed on $\| \ \|$ implies the existence of a constant $\eta > 0$ such that $|\psi(\theta) - \tilde{\varphi}(\theta)| < \eta$ for all θ in $[t_o, t_o + \tau)$ implies

$$\|\psi | [t_o, t_o + \tau) - \tilde{\varphi} | [t_o, t_o + \tau)\| < \delta \quad .$$

Property (2) in turn together with (2.3) allows us to conclude that $|\psi(\theta) - \tilde{\varphi}(\theta)| < \eta$ for θ in α_t implies that $\|\psi(\alpha_t) - \tilde{\varphi}(\alpha_t)\| < \delta$. Clearly (2.3) and (2.4) imply we can choose $\tau_2 > 0$ such that $|\psi(\alpha_t)(\theta) - \tilde{\varphi}(\alpha_t)(\theta)| < \eta$ for all t in $[t_o, t_o + \tau_2]$ and θ in α_t. Hence for t in $[t_o, t_o + \tau_2]$ we have

$$|G(t, \psi(\alpha_t)) - G(t, \tilde{\varphi}(\alpha_t))| < \frac{1}{2} \quad ,$$

for arbitrary ψ in $S(b_1, \tau_2)$. Therefore,

$$|G(t, \psi(\alpha_t)) - M_o + M_o - G(t, \tilde{\varphi}(\alpha_t))| < \frac{1}{2} \quad ,$$

and

$$|G(t, \psi(\alpha_t))| < M_o + |M_o - G(t, \tilde{\varphi}(\alpha_t))| + \frac{1}{2} \quad .$$

Clearly then we may choose $\tau_1 > 0$ such that $\tau_1 < \tau_2$, $(M_o + 1)\tau_1 < b_1$, $|M_o - G(t, \tilde{\varphi}(\alpha_t))| < \frac{1}{2}$, and

$$|G(t, \psi(\alpha_t))| \leq M_o + 1$$

for all t in $[t_o, t_o + \tau_1]$ and ψ in $S(b_1, \tau_1)$.

Now consider an operator T defined on $S(b_1, \tau_1)$ by the formula

$$\begin{aligned} &T(\psi) \mid (-\infty, t_o] = \psi \mid (-\infty, t_o], = \phi \mid (-\infty, t_o] \\ &T(\psi)(t) = \varphi(t_o) + \int_{t_o}^{t} G(s, \psi(\alpha_s))ds, \end{aligned} \tag{2.5}$$

for t in $[t_o, t_o + \tau_1]$. For an arbitrarily specified function ψ_1 in $S(b_1, \tau_1)$, our continuity conditions on G imply that for arbitrary $\epsilon > 0$ we can choose $\delta > 0$ such that ψ in $S(b_1, \tau_1)$ and $\|\psi_1 - \psi\| < \delta$ implies

$$|G(s, \psi_1(\alpha_s)) - G(s, \psi(\alpha_s))| < \frac{\epsilon}{\tau_1} .$$

Hence for all t in $[t_o, t_o + \tau_1]$ and $\|\psi_1 - \psi\| < \delta$, we have

$$\begin{aligned} |T(\psi_1)(t) - T(\psi)(t)| &\leq \int_{t_o}^{t} |G(s,\psi_1(\alpha_s)) - G(s, \psi(\alpha_s))|ds \\ &< \frac{\epsilon}{\tau_1} (t - t_o) \leq \epsilon, \end{aligned}$$

which implies $\|T(\psi_1) - T(\psi)\| < \epsilon$ and establishes the continuity of T on $S(b_1, \tau_1)$. We observe that

$$|T(\psi)(t) - \varphi(t_o)| \leq \int_{t_o}^{t} |G(s,\psi(\alpha_s))|ds \leq (M_o + 1)\ (t - t_o) < b_1 .$$

Also we have for arbitrary t_1 and t_2 in $[t_o, t_o + \tau_1]$ and ψ in $S(b_1,\tau_1)$ that

$$|T(\Psi)(t_1) - T(\Psi)(t_2)| \leq \int_{t_1}^{t_2} |G(s, \Psi(\alpha_s))|ds \leq (M_o + 1)|t_2 - t_1|.$$

Hence it follows that T maps $S(b, \tau_1)$ into itself and from the Schauder fixed point theorem we may conclude the existence of a function x in $S(b_1, \tau_1)$ such that $T(x) = x$. That is, x is such that $x|(-\infty, t_o] = \Phi|(-\infty, t_o]$ and

$$x(t) = \Phi(t_o) + \int_{t_o}^{t} G(s, x(\alpha_s))ds,$$

for t in $[t_o, t_o + \tau_1]$. Taking right hand derivatives it is clear that

$$\dot{x}(t) = G(t, x(\alpha_t))$$

for t in $[t_o, t_o + \tau_1]$. Thus x is a solution of equation (2.1) on $[t_o, t_o + \tau_1]$ and the proof of our theorem is complete.

Let us now proceed by stating and proving an existence and uniqueness theorem for hereditary equations which reduces to a standard Picard-Lindelof type theorem for ordinary differential equations.

Theorem 2.2: *Let $G(t, \Psi(\alpha_t))$ be a right-continuous function of t in $[t_o, t_o + \tau_o)$, $\tau_o > 0$, for Ψ in $U_{t_o}(B)$. Suppose that for arbitary Ψ and Ψ' in $U_{t_o}(B)$ we have that*

$$|G(t, \Psi(\alpha_t)) - G(t, \Psi'(\alpha_t))| \leq K(t)\,||\Psi(\alpha_t) - \Psi'(\alpha_t)|| \,, \tag{2.6}$$

where $K(t)$ is an integrable function of t. Then for each function ϕ in $U_{t_o}(B)$, there exists a unique solution x of equation (2.1) defined on $[t_o, t_o + \tau_1)$, for some $\tau_1 > 0$, and corresponding to ϕ at t_o. Furthermore, solutions x depend continuous on their initial data.

Proof. The existence of a solution of equation (2.1) corresponding to ϕ at t_o and defined on some interval $[t_o, t_o + \tau_2)$, $\tau_2 > 0$, follows from Theorem 1. Hence we have only to establish uniqueness and continuous dependence on initial data.

Suppose x_1 and x_2 are two solutions of (2.1) corresponding to admissable initial functions ϕ_1 and ϕ_2. Then we have

$$x_1(t) - x_2(t) = \phi_1(t_o) - \phi_2(t_o) + \int_{t_o}^{t} (G(\tau, x_1(\alpha_\tau)) - G(\tau, x_2(\alpha_\tau)))\,d\tau,$$

for t in some interval $[t_o, t_o + \tau_1]$, $\tau_1 > 0$. Hence using condition (2.6) we have that

$$|x_1(t) - x_2(t)| \leq |\phi_1(t_o) - \phi_2(t_o)| + \int_{t_o}^{t} K(\tau)||x_1(\alpha_\tau) - x_2(\alpha_\tau)||d\tau$$

$$\leq g(t) + \int_{t_o}^{t} K(\tau)\, ||x_1(\alpha_\tau \cap [t_o,\tau]) - x_2(\alpha_\tau \cap [t_o,\tau])||d\tau ,$$

where

$$g(t) = |\phi_1(t\ ) - \phi_2(t\ )| + ||\phi_1(\overleftarrow{\alpha}_{t_o}) - \phi_2(\overleftarrow{\alpha}_{t_o})|| \int_{t_o}^{t} K(\tau)\, d\tau$$

By our conditions on $|| \ ||$ it follows that there exists a constant b such that

$$|| x_1(\alpha_t \cap [t_o, t]) - x_2(\alpha_t \cap [t_o,t])|| \leq bg(t) + b\int_{t_o}^{t} K(\tau)||x_1(\alpha_\tau \cap [t_o,t]) - x_2(\alpha_\tau \cap [t_o,t])|| d\tau .$$

Now by Gronwall's lemma it follows that

$$|| x_1(\alpha_t \cap [t_o, t]) - x_2(\alpha_t \cap [t_o, t])|| \leq bg(t) + b^2\int_{t_o}^{t} K(\tau)g(\tau)\exp\left(b\int_{t_o}^{t} K(s)ds\right)d\tau . \tag{2.8}$$

If $\phi_1 = \phi_2$ it is clear that g is identically zero, which in turn implies $x_1 = x_2$ and the uniqueness of the solution of (2.1) corresponding to a specified initial function. It is also clear that g can be made as small as we like over any finite interval $[t_o, t]$ if $||\phi_1(\overleftarrow{\alpha}_{t_o}) - \phi_2(\overleftarrow{\alpha}_{t_o})||$ is taken sufficiently small, so continuity with respect to initial data follows from (2.8) and our theorem is proved.

At this point let us observe the situation when the function space $Q((-\infty, t_o], R^n)$ used in the definition of $U_{t_o}(B)$ is taken to be single function. In this case continuity with respect to initial data is only established in a degenerate, sense, but we, of course, still have the establishment of a unique solution.

Theorem 2.2 can also be proved using the contraction principle. To see this let us specify a function φ on R and define $U(\tau)$ to be the set of all functions ψ defined on $(-\infty, t_o + \tau]$, $\tau > 0$, such that $\psi|(-\infty, t_o] = \phi|(-\infty, t_o]$ and $\psi|[t_o, t_o + \tau]$ is contained in $C([t_o, t_o + \tau], B)$. With each element ψ in $U(\tau)$ let us associate the norm

$$\|x\| = \sup_{s\in[t_o,\, t_o+\tau]} \|x(\alpha_s)\| \quad .$$

We define a mapping T on $U(\tau)$ by the formula

$$T\psi|(-\infty, t_o] = \phi_1(-\infty, t_o]$$

$$T\psi(t) = \varphi(t_o) + \int_{t_o}^{t} G(s, \psi(\alpha_s))ds, \quad t \text{ in } [t_o, t_o + \tau].$$

For each pair ψ_1, ψ_2 in $U(\tau)$ we have

$$|T\psi_1(t) - T\psi_2(t)| \leq \int_{t_o}^{t} K(s)\|\psi_1(\alpha_s) - \psi_2(\alpha_s)\|ds, \quad \theta \in [t_o, t_o + \tau],$$

and there must exist $b > 0$ such that

$$\|T\psi_1(\alpha_t) - T\psi_2(\alpha_t)\| \leq b \int_{t_o}^{t} K(s)\|\psi_1(\alpha_s) - \psi_2(\alpha_s)\|ds$$

Hence we conclude that

$$\|T\psi_1 - T\psi_2\| \leq \left(b \int_{t_o}^{t} K(s)ds\right)\|\psi_1 - \psi_2\|,$$

and certainly we may choose $\tau = \tau_1$, so that

$$b \int_{t_o}^{t_o + \tau_1} K(s)ds = \beta < 1.$$

This, of course, implies T is a contraction on $U(\tau_1)$, and we may conclude that there is a unique point ψ^* in $U(\tau_1)$ such that $T(\psi^*) = \psi^*$. Since a function in $U(\tau_1)$ is a solution of (12) if and only if it is a fixed point under T, we have established existence and uniqueness. Continuous dependence on initial data is, as we have demonstrated, a straightforward application of Gronwall's lemma.

An important feature of establishing Theorem 2 by setting up a contraction mapping, is that it reveals a systematic way of constructing a solution x of equation (12) starting with any element ψ in $U(\tau_1)$. Specify

$$x = \lim_{n \to \infty} T^n(\psi),$$

and if x is estimated by $T^n(\psi)$ for some finite n, one can easily compute that

$$\|x - T^n(\psi)\| \leq \frac{\beta^n}{1 - \beta} \|T(\psi) - \psi\| .$$

For a result analogous to the Caratheodory theorem for ordinary differential equations one may state the following theorem.

Theorem 2.3: For Ψ in U_{t_0} (B) let $G(t, \Psi(\alpha_t))$ be a continuous function of $\Psi(\alpha_t)$ uniformly for t in $[t_0, t_0 + \tau_0)$, and let $G(t, \Psi(\alpha_t))$ be a measureable function of t on $[t_0, t_0 + \tau_0)$. If there exists a Lebesque integrable function m such that $|G(t, \Psi(\alpha_t))| \leq m(t)$ for all t in $[t_0, t_0 + \tau)$ and Ψ in U_{t_0} (B), then for ϕ in U_{t_0} (B) there exists $\tau_1 > 0$ such that equation (2.1) has a solution x in the extended sense on $[t_0, t_0 + \tau_1)$ which corresponds to ϕ at t_0.

Note: A solution of (2.1) in the extended sense is a function x such that $x(\overleftarrow{\alpha}_{t_0}) = \phi(\overleftarrow{\alpha}_{t_0})$ and on $[t_0, t_0 + \tau)$ is absolutely continuous and satisfied the equation.

$$x(t) = \phi(t_0) + \int_{t_0}^{t} G(\tau, x(\alpha_\tau))\, d\tau \tag{2.7}$$

Proof. Let b_1 be such that $\{y: |y - \phi(t_0)| \leq b_1\} \subset B$, and let $S(b_1, \tau)$ be the set of all functions Ψ defined on $(-\infty, t_0 + \tau]$, $\tau < \tau_0$, satisfying the following conditions:

$$\Psi|(-\infty, t_0] = \Phi|(-\infty, t_0],$$

$$|\Psi(t) - \phi(t_0)| \leq b,$$

for all t in $[t_0, t_0 + \tau]$. It is clear that $S(b_1, \tau)$ is convex and closed in the norm topology determined by the functional

$$\| \Psi \| = \sup\{|\Psi(t)| : t \text{ in } [t_0, t_0 + \tau_0]\}.$$

We define the operator T on $S(b_1, \tau)$ by the formula

$$T(\psi|(-\infty, t_o]) = \varphi\,|(-\infty, t_o]$$

$$T(\psi)(t) = \varphi(t_o) + \int_{t_o}^{t} G(s, \psi(\alpha_s))ds,$$

for t in $(t_o, t_o + \tau]$. Since G is by hypothesis measurable in s, the operator T is continuous on $S(b_1, \tau)$. Let $\{\psi_n\}$ be any sequence of functions in $S(b_1, \tau)$ converging to ψ. Clearly we have that

$$|T(\psi)(t) - T(\psi_n)(t)| \leq \int_{t_o}^{t} |G(s, \psi(\alpha_s)) - G(s, \psi(\alpha_s))|ds.$$

Now by hypothesis $G(s, \psi(\alpha_s))$ is continuous in $\psi(\alpha_s)$ uniformly for s in $[t_o, t_o + \tau]$, so

$$|G(s, \psi(\alpha_s)) - G(s, \psi_n(\alpha_s))| \to 0$$

uniformly for s in $[t_o, t_o + \tau]$ as $n \to \infty$. Furthermore, the functions $|G(s, \psi(\alpha_s)) - G(s, \psi_n(\alpha_s))|$ are measurable in s and such that

$$|G(s, \psi(\alpha_s)) - G(s, \psi_n(\alpha_s))| \leq 2m(s)$$

for s in $[t_o, t]$, $t < t_o + \tau_o$. Hence by Lebesgue's theorem on majorized sequences we have that

$$\int_{t_o}^{t} |G(s, \psi(\alpha_s)) - G(s, \psi_n(\alpha_s))|ds \to 0$$

as $n \to \infty$, and it follows that T is continuous in $S(b_1, \tau)$. Clearly $\tau_1 > 0$ may be chosen such that

$$|T(\psi)(t) - \varphi(t_o)| \leq \int_{t_o}^{t} m(s)ds \leq b_1$$

for all t in $[t_o, t_o + \tau_1]$ and ψ in $S(b_1, \tau_1)$, so

$$T(S(b_1, \tau_1)) \subset S(b_1, \tau_1).$$

Also since $\int_{t_o}^{t} m(s)ds$ is a uniformly continuous function on $[t_o, t_o + \tau_1]$, it follows that for every $\epsilon > 0$ we may choose a number $\eta = \eta(\epsilon) > 0$ such that t_1, t_2 in $[t_o, t_o + \tau_1]$ and $|t_1 - t_2| < \eta$ imply

$$|T(\psi)(t_2) - T(\psi)(t_1)| \leq \int_{t_1}^{t_2} m(s)ds < \epsilon \quad .$$

Thus $T(S(b_1, \tau_1))$ is a equicontinuous family of functions and it follows from the Arzela-Ascoli Theorem that $T(S(b_1, \tau_1))$ is conditionally compact. But, of course, this implies T is completely continuous and we may employ the Schauder fixed point theorem to conclude the existence of a fixed point under T. This fixed point is obviously absolutely continuous and satisfies equation (2.7), so the proof of our theorem is complete.

3. Proper Hereditary Dependence

Reconsidering at this point the hypotheses stated in our existence and uniqueness theorems thus far, we observe that our direct structural requirements on the vector functional G are rather strong whereas no specified structure is required in our initial functions. We shall now formulate our class of differential equations of hereditary type in such a way as to allow us to draw more on the structure of initial functions in questions of existence and uniqueness of solutions and require less direct structural assumptions on our functional forms. To proceed, let us specify a lag function α and define the set of functions

$$Q_{\alpha, t_0}(B) = \{x(\alpha_t): \ t \geq t_0, \ x \ \text{in} \ U_{t_0}(B)\} \quad .$$

We observe that in equation (2.1), G can be thought of as a function mapping $[t_0, t_0 + \tau_0) \times Q_{\alpha, t_0}(B)$ into R^n. A little investigation will quickly reveal, however, that the function space from which the second argument of G is taken is extremely awkward indeed when we try to introduce a reasonable notion of continuity. It is not at all immediately clear what structure should be used to define a topology. We shall take our clue, however, from a generalized notion of translation.

Let $\omega \subset (-\infty, 1]$ be a specified fixed element in Ω. We shall call a lag function α proper if for some chosen set ω we have:

(1) For each t in R there exists a continuous mapping h_t of ω into α_t which preserves order.

(2) The family of functions $\{h_t: t \text{ in } R\}$ depend continuously on t. That is, for every $\varepsilon > 0$ there exists a $\delta = \delta(t, \varepsilon) > 0$ such that $|t - \tau| < \delta$ implies

$$\|h_t - h_\tau\| = \sup\{|h_t(\theta) - h_\tau(\theta)|: \theta \text{ in } \omega\} < \varepsilon .$$

Let us define $x: R \to B$, and for every t in R let the mapping $H_t x$ on ω be defined by the formula

$$H_t x(\theta) = x(h_t(\theta)), \quad \theta \text{ in } \omega . \tag{3.1}$$

Since we have a well defined norm on $Q(\omega, B)$, for two functions $x: R \to B$ and $y: R \to B$ we can express the distance ρ between two functions $x(\alpha_t)$ and $y(\alpha_\tau)$ by the formula

$$\rho(x(\alpha_t), y(\alpha_\tau)) = \|H_t x - H_\tau y\| . \tag{3.2}$$

The notion of continuity imposed by this metric seems most natural and specializes for particular cases to the notion of continuity traditionally employed. For example, consider equation (1.4) where $\alpha_t = [t - 1, t]$ for all t in R. We may select $\omega = [-1, 0]$ and define

$$h_t(\theta) = t + \theta, \quad \theta \text{ in } \omega , \tag{3.3}$$

which is, of course, a simple translation of the type usually employed. We are also free, however, to introduce a weighting factor such as illustrated by defining h_t by the formula

$$h_t(\theta) = t + \theta^{\xi}, \ \theta \text{ in } \omega, \ \xi \geq 0. \tag{3.4}$$

Classes of mappings of the type $\{h : t \text{ in } R\}$ are denoted as translation classes.

Let us now proceed to formulate a class of differential equations with hereditary dependence based on proper lag functions and the topology of $Q(\omega,B)$. Specifically, let us define a function $F: [t_o, t_o + \tau_o) \times Q(\omega,B) \to R^n$, $\tau_o > 0$, and consider the functional relation

$$\dot{x}(t) = F(t, H_t x), \quad t \geq t_o. \tag{3.5}$$

As before, the initial data problem is set up by specifying a n-vector function ϕ on R. A function x such that $x(\overleftarrow{\alpha}_{t_o}) = \phi(\overleftarrow{\alpha}_{t_o})$ and which satisfies equation (3.5) on $[t_o, t_o + \tau)$ for some $\tau > 0$ is referred to a a solution corresponding to ϕ at t_o.

To illustrate our structure further we note that test sets ω may be defined by the formulas

$$\omega = \{0\},$$

$$\omega = \{0, -\tau_1, -\tau_2, \ldots, \tau_n\},$$

$$\omega = \{0, -1, -2, \ldots \},$$

$$\omega = [-1, 0],$$

and

$$\omega = (-\infty, 0],$$

respectively for equations (1.1), (1.2), (1.3), (1.4), and (1.5), and in each case h_t and H_t may be specified by the expressions

$$h_t(\theta) = t + \theta, \quad \theta \text{ in } \omega,$$

and

$$H_t x(\theta) = x(h_t(\theta)) = x(t + \theta), \quad \theta \text{ in } \omega.$$

For equation (1.6) we may take

$$\omega = \{\tfrac{1}{2}, 1\}$$

$$h_t(\theta) = \theta t, \quad \theta \in \omega,$$

$$H_t x(\theta) = x(h_t(\theta)) = x(\theta t), \quad \theta \in \omega$$

and for equation (1.8),

$$\omega = [0, 1]$$

$$h_t(\theta) = (1 - \theta)\left(\frac{t-1}{t+1}\right) + \theta t, \quad \theta \text{ in } \omega$$

$$H_t x(\theta) = x(h_t(\theta)) = x((1 + \theta)\left(\frac{t-1}{t+1}\right) + \theta t), \quad \theta \text{ in } \omega;$$

We remark in passing that general hereditary equations such as we have defined are not necessarily autonomous even if t does not occur explicitly as a variable. That is, a hereditary equation of the form

$$\dot{x}(t) = \Theta(H_t x), \qquad t \geq t_o, \tag{3.6}$$

is not necessarily autonomous. In the cases where lag functions such as those in examples (1.1), (1.2), (1.3), (1.4), and (1.5) are involved, however, equation (3.6) is autonomous.

An extended concept of hereditary dependence worthy of mention is incorporated in the notion of a matrix lag function. That is, we could consider a matrix α of lag functions α_{ij}, $i, j = 1,\ldots,n$. Such a construction would be useful if one wishes to distinguish between the hereditary dependence of the various components in a system. In particular one may wish to consider a system of the form

$$\dot{x}_i(t) = F_i(t, H_t^{i1} x_1, H_t^{i2} x_2, \ldots, H_t^{in} x_n), \quad i = 1, 2, \ldots, n,$$

where, of course, the operators H^{ij} are determined by the lag functions α_{ij} in the manner previously discussed. Clearly this system can be written more compactly as

$$\dot{x}(t) = F(t, H_t x),$$

where

$$F(t, H_t x) = (F_1(t, H_t^1 x), \ldots, F_n(t, H_t^n x))$$

and

$$H_t^i x = (H_t^{i1} x, H_t^{i2} x, \ldots, H_t^{in} x).$$

A norm on the elements $H_t x$ can obviously be constructed by considering a matrix norm for matrices of the form $(\|H_t^{ij} x\|)$.

It may also be convenient to consider split hereditary dependence. That is, it may be an advantage to consider equations of the form

$$\dot{x}(t) = F(t, H_t x, H'_{\tau(t)} x),$$

where H_t and $H'_{\tau(t)}$, $\tau(t) \leq t$, correspond to different lag functions. This will mainly be true when the dependence of F on the arguments $H_t x$ and $H'_{\tau(t)} x$ is distinguishable. In particular the dependence of F on $H'_{\tau(t)} x$ might be only continuous whereas the dependence on $H_t x$ might be linear. To illustrate further consider an equation of the form

$$\dot{x}(t) = f(t, x(t - 1), x(t))$$

where $f(t, u, v)$ is continuous on $R \times R^n \times R^n$ and linear in v. A straightforward argument will show that corresponding to each continuous initial function specified on $[-1, 0]$ that this equation has a unique solution existing for all $t \geq 0$.

4. More Existence and Uniqueness Theorems

Let us now state theorems analogous to Theorems 2.1, 2.2, and 2.3 where α is assumed to be a proper lag function.

Theorem 4.1: If $F(t,H_t\psi)$ is a continuous function of $H_t\psi$ for t, ψ in $[t_o,t_o+\tau_o) \times U_{t_o}(B)$, $\tau_o > 0$, and right-continuous in t for all ψ in $U_{t_o}(B)$, then for each ϕ in $U_{t_o}(B)$ there exists $\tau_1 > 0$ such that equation (3.5) has a solution x on $[t_o,t_o+\tau_1)$ corresponding to ϕ at t_o.

Theorem 4.2: Let $F(t,H_t\psi)$ be a right-continuous function of t on $[t_o,t_o+\tau_o)$, $\tau_o > 0$, for ψ in $U_{t_o}(B)$. Suppose that for arbitrary ψ and ψ' in $U_{t_o}(B)$ we have that

$$| F(t,H_t\psi) - F(t,H_t\psi') | \leqq K(t) \| H_t\psi - H_t\psi' \| ,$$

where $K(t)$ is an integrable function of t. Then for each function ϕ in $U_{t_o}(B)$ there exists a unique solution x of equation (3.5) defined on $[t_o,t_o+\tau_1)$, for some $\tau_1 > 0$, and corresponding to ϕ at t_o. Furthermore, solutions x depend continuously on their initial data.

<u>Theorem 4.3:</u> <u>For</u> ψ <u>in</u> $U_{t_o}(B)$ <u>let</u> $F(t,H_t\psi)$ <u>be a continuous function of</u> $H_t\psi$ <u>for</u> t <u>in</u> $[t_o,t_o+\tau_o)$, <u>and let</u> $F(t,H_t\psi)$ <u>be a measurable function of</u> t <u>on</u> $[t_o,t_o+\tau_o)$. <u>If there exists a Lebesgue integrable function</u> m <u>such that</u> $|G(t,H_t\psi)| \leq m(t)$ <u>for all</u> t <u>in</u> $[t_o,t_o+\tau)$ <u>and</u> ψ <u>in</u> $U_{t_o}(B)$, <u>then for</u> ϕ <u>in</u> $U_{t_o}(B)$ <u>there exists</u> $\tau_1 > 0$ <u>such that equation (3.5) has a solution</u> x <u>in the extended sense on</u> $[t_o,t_o+\tau_1)$ <u>which corresponds to</u> ϕ <u>at</u> t_o.

Theorems 4.1 , 4.2 , and 4.3 can be proved by essentially the same arguments used respectively for Theorems 2.1 , 2.2 , and 2.3.

For ω in Ω and B an open connected set in R^n, let us define $P(\omega,B)$ to be the set of all functions $\phi : \omega \to B$ which are right-continuous. For ψ in $P(R,R^n)$, $H_t\psi$ is said to be a right-continuous function of t on an interval in R if $H_t\psi$ is a continuous function of t except on a closed discrete set and for t in this exceptional discrete set $H_t\psi$ is continuous from the right and $\lim_{s\to t-} H_s\psi$ exists in a function in $P(\omega,R^n)$.

<u>Theorem 4.4:</u> <u>Let</u> F <u>be continuous on</u> $[t_o,t_o+\tau_o) \times P(\omega,B)$. <u>Suppose that for all</u> (t,ψ) <u>in</u> $[t_o,t_o+\tau_o) \times U_{t_o}(B)$, $H_t\psi$ <u>is right-continuous.</u> <u>Then for each</u> ϕ <u>in</u> $U_{t_o}(B)$ <u>there exists</u> $\tau_1 > 0$ <u>such that equation (3.5) has a solution</u> x <u>on</u> $[t_o,t_o+\tau_1)$ <u>corresponding to</u> ϕ <u>at</u> t_o.

<u>Proof.</u> Let ψ be an arbitrary function in $U_{t_o}(B)$ and let t^* be an arbitrary but fixed point in $[t_o, t_o + \tau_o)$. Since F is continuous on $[t_o, a) \times P(\omega, B)$, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, t^*) > 0$ such that $\max\{|t - t^*|, ||\xi - H_{t^*}\psi||\} < \delta$ implies

$$|F(t, \xi) - F(t^*, H_{t^*}\psi)| < \varepsilon .$$

If we suppose t^* to be a point of continuity of $H_t\psi$, then there exists a positive number $\delta_1 = \delta_1(\delta, t^*) < \delta$ such that $|t - t^*| < \delta_1$ implies $||H_t\psi - H_{t^*}\psi|| < \delta$. Hence $|t - t^*| < \delta_1$ implies $|F(t,H_t\psi) - F(t^*,H_{t^*}\psi)| < \varepsilon$, and it follows that t^* is a point of continuity of $F(t, H_t\psi)$ as a function of t. Suppose t^* is a point of discontinuity of $H_t\psi$ and let $H_{t^*-}\psi = \lim_{t \to t^*-} H_t\psi$. Since $H_{t^*-}\psi$ is contained in $P(\omega, B)$ it follows that for every $\varepsilon > 0$ there exists a number $\eta = \eta(\varepsilon, t^*) > 0$ such that $\max\{|t - t^*|, ||\xi - H_{t^*-}\psi||\} < \eta$ implies

$$|F(t, \xi) - F(t^*, H_{t^*-}\psi)| < \varepsilon .$$

But there exists $\eta_1 > 0$ such that $\eta_1 = \eta_1(\varepsilon, t^*) < \eta$ and $0 < t^* - t < \eta_1$ implies $||H_t\psi - H_{t^*-}\psi|| < \eta$. Hence $0 < t^*-t < \eta_1$ implies $|F(t, H_t\psi) - F(t^*, H_{t^*-}\psi)| < \varepsilon$ and it follows that

$$\lim_{t \to t^*-} F(t, H_t\psi) = F(t, H_{t^*-}\psi)$$

Thus $F(t, H_t\psi)$ has the same points of continuity and discontinuity as $H_t\psi$ and the points of discontinuity are of the same type. It follows, of course, that $F(t, H_t\psi)$ is a right continuous function of t on $[t_o, t_o + \tau_o)$. To complete the proof of this Theorem we have only to apply Theorem 4.1.

Corollary: If in Theorem 4.4 we replace $P(\omega,B)$ by $C(\omega,B)$ and replace "right-continuous" by "continuous", whereever it occurs, then we still have a true theorem.

Now for ω in Ω and B an open and connected subset of R^n let us define

$$H(\omega,B) = \{ H_t\psi : \psi \text{ in } U_{t_o}(B), \ t \geq t_o \} . \tag{4.1}$$

In all our existence theorems thus far our hypotheses have included an explicit continuity condition for F or G with respect to t as it occurs in both the first and second argument. In our next theorem we shall not include such a condition but shall instead require continuity on $[t_o, t_o + \tau_o) \times H(\omega, B)$ only. However, we shall first need to introduce the notion of uniform hereditary dependence.

Consider two functions ψ and ϕ in $U_{t_o}(B)$ and an interval $[t_1, t_2]$ in R. We shall denote the norm $|| \; ||$ of the difference $H_t\psi - H_t\phi$ taken with respect to the restricted domain $h_t^{-1}(\alpha_t \cap [t_1, t_2])$ by

$$|| H_t\psi - H_t\phi ||_{t_1, t_2}$$

With respect to the particular restriction $h_t^{-1}(\alpha_t \cap (-oo, \tau])$ we shall use the notation

$$|| H_t\psi - H_t\phi ||_{\leftarrow \tau}$$

$\overset{\leftarrow \tau}{H_t\psi}$ denotes the restriction of $H_t\psi$ to $h_t^{-1}(\alpha_t \cap (-oo, \tau])$.

If for each ψ in $U_{t_o}(B)$ $\overset{\leftarrow t_o}{H_t \psi}$ is uniformly continuous in t for $t \geqq t_o$, then equation (3.5) is said to have <u>uniform hereditary dependence</u> (with respect to $U_{t_o}(B)$).

<u>Theorem 4.5</u> : <u>Let</u> F <u>be continuous on</u> $[t_o, t_o + \tau_o) \times H(\omega, B)$ <u>and suppose equation (3.5) has uniform hereditary dependence. Then for each</u> ϕ <u>in</u> $U_{t_o}(B)$ <u>there exists</u> $\tau_1 > 0$ <u>such that equation (3.5) has a solution</u> x <u>on</u> $[t_o, t_o + \tau_1)$ <u>corresponding to</u> ϕ <u>at</u> t_o. <u>Furthermore</u>, x <u>is continuously differentiable on</u> $[t_o, t_o + \tau_1)$ <u>from the right</u>.

<u>Proof</u> : Choosing ψ in $U_{t_o}(B)$ we observe that the continuity of F on $[t_o, t_o + \tau_o) \times H(\omega, B)$ implies that for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, t)$ such that if $\max \{|t - \tau|, \| H_t\psi - H_\tau\psi \| \} < \delta$, then

$$|F(t, H_t\psi) - F(\tau, H_\tau\psi)| < \varepsilon .$$

Now there exists a constant b such that

$$\| H_t\psi - H_\tau\psi \| \leqq \| H_t\psi - H_\tau\psi \|_{\leftarrow t_o} \tag{4.2}$$

$$+ b \max \{|\psi(h_t(\theta)) - \psi(h_\tau(\theta))| : h_t(\theta) \text{ in } \alpha_t \setminus \overleftarrow{\alpha}_{t_o},$$

$$h_\tau(\theta) \text{ in } \alpha_t \setminus \overleftarrow{\alpha}_{t_o}, \ t,\tau \text{ in } [t_o, t_o + \tau_o]\}.$$

Since ψ is uniformly continuous on $[t_o, t_o + \tau_1]$, $0 < \tau_1 < \tau_o$, there exists $\eta>0$ such that $\| h_t - h_\tau \| < \eta$ implies the second term on the right hand side of (4.2) is $< \frac{\delta}{2b}$. On the other hand using the continuity of h_t and our condition of uniform hereditary dependence we can choose $\mu>0$ such that $|t - \tau| < \mu$ implies

$$\| h_t - h_\tau \| < \eta \qquad \text{and} \qquad \| H_t\psi - H_\tau\psi \|_{\leftarrow t_o} < \frac{\delta}{2} .$$

Then choosing τ such that $|t - \tau| < \min \{\delta, \mu\}$ implies $\| H_t\psi - H_\tau\psi \| < \delta$ and we have in turn that

$$|F(t, H_t\psi) - F(\tau, H_\tau\psi)| < \varepsilon .$$

We conclude, therefore, that $F(t, H_t\psi)$ is a continuous function of t and our theorem follows as a consequence of Theorem 4.1.

<u>Corollary</u> : <u>Let</u> F <u>be continuous on</u> $[t_o, t_o + \tau_o) \times C(\omega, B)$. <u>Then for any uniformly continuous function</u> ϕ <u>in</u> $C((-\infty, t_o], B)$ <u>there exists a number</u> $\tau_1 > 0$ <u>such that equation (3.5) has a solution</u> x <u>defined on</u> $[t_o, t_o + \tau_o)$ <u>corresponding to</u> ϕ <u>at</u> t_o. <u>Furthermore</u>, x <u>is continuously differentiable on</u> $[t_o, t_o + \tau_1)$ <u>from the right</u>.

<u>Proof</u>. Let $Q((-\infty, t_o], R^n)$ used in the definition of $U_{t_o}(B)$ consist of the single function ϕ. Since ϕ is uniformly continuous, we have that given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\| h_t - h_\tau \| < \delta$ implies $|\phi(h_t(\theta)) - \phi(h_\tau(\theta))| < \varepsilon$ for all θ in $h^{-1}((-\infty, t_o])$. But by the continuity of h_t we may choose $\eta > 0$ such that $|t - \tau| < \eta$ implies $\| h_t - h_\tau \| < \delta$ and consequently implies $|\phi(h_t(\theta)) - \phi(h_\tau(\theta))| < \varepsilon$. Hence we may conclude that we have uniform hereditary dependence and apply Theorem 4.5 to complete our proof.

<u>Theorem 4.6</u>: <u>Let</u> F <u>be continuous on</u> $[t_o, t_o + \tau_o) \times P(\omega, B)$ <u>where</u> ω <u>is compact and</u> $\| \; \|$ <u>is defined by the formula</u>

$$\| \phi \| = \left(\int_\omega |\phi(\theta)|^2 d\theta + |\phi(t)|^2 \right)^{1/2},$$

$t = \max \{\theta : \theta \text{ in } \omega\}$. <u>Then for any function</u> ϕ <u>in</u> $P((-\infty, t_o], B)$

<u>there exists a number</u> $\tau_1 > 0$ <u>such that equation (3.5) has a solution</u> x <u>defined on</u> $[t_o, t_o + \tau_o)$ <u>corresponding to</u> ϕ <u>at</u> t_o. <u>Furthermore</u>, x <u>is continuously differentiable on</u> $[t_o, t_o + \tau_1)$ <u>from the right</u>.

<u>Proof</u>. Let $U_{t_o}(B)$ be defined so that $U((-\infty, t_o], R^n)$

$U((-\infty, t_o], R^n) = P((-\infty, t_o], B)$. the continuity of F on $[t_o, t_o + \tau_o) \times P(\omega, B)$ implies that for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, t)$ such that if $\max \{ |t - \tau|, \| H_t\psi - H_\tau\psi \| \} < \delta$, then

$$|F(t, H_t\psi) - F(\tau, H_\tau\psi)| < \varepsilon .$$

By the right-continuity of ψ and the compactness of ω we can establish the existence of a number $\eta = \eta(\delta) > 0$ such that $\| h_t - h_\tau \| < \eta(\delta)$ implies $\| H_t\psi - H_\tau\psi \| < \delta$. The continuous dependence of the function h_t on t implies that we may choose $\mu = \mu(\eta, t)$ such that $|t - \tau| < \mu$ implies $\| h_t - h_\tau \| < \eta$. Therefore, choosing τ such that $|t - \tau| < \min \{\delta, \mu\}$ implies $|F(t, H_t\psi) - F(\tau, H_\tau\psi)| < \varepsilon$, and we may conclude that $F(t, H_t\psi)$ is a continuous function of t. Hence our theorem follows as a consequence of Theorem 4.1.

Our next theorem is a generalization of Theorem 4.2 which draws attention to the special nature of hereditary differential equations. Clearly the statement and results of this theorem would not even be meaningful for differential equations in a Banach space generally.

Theorem 4.7 : Let $G(t,\psi(\alpha_t))$ be a right continuous function of t for all t in $[t_o, t_o + \tau_o)$, $\tau_o > 0$, and all ψ in $U_{t_o}(B)$. For fixed $\eta > 0$ let $\beta_t = \alpha_t \cap \alpha_{t-\eta}$, $\gamma_t = \overline{\alpha_t \setminus \alpha_{t-\eta}}$, and for each ψ in $U_{t_o}(B)$, let $\mathcal{H}(t,\psi(\alpha_t))$ be an n-vector function continuous in $\psi(\beta_t)$ uniformly in t and measurable in t on $[t_o, t_o + \tau_o)$. For arbitrary ψ ψ' in $U_{t_o}(B)$, let

$$|G(t,\psi(\alpha_t)) - G(t,\psi'(\alpha_t))| \leq |\mathcal{H}(t,\psi(\beta_t)) - \mathcal{H}(t,\psi'(\beta_t))| \tag{4.2}$$

$$+ K(t) \, || \psi(\gamma_t) - \psi'(\gamma_t) || ,$$

where $K(t)$ is a positive integrable function. If there exists an integrable function m on $[t_o, t_o + \tau_o)$ such that $|\mathcal{H}(t, \psi(\beta_t))| \leq m(t)$, then for each ϕ in $U_{t_o}(B)$, there exists a unique solution x of equation (2.1) *defined on* $[t_o,t_o + \tau_1]$, *for some* $\tau_1 > 0$, *and corresponding to* ϕ *at* t_o. *Furthermore, solutions* x *depend continuously from the right on their initial data.*

Proof. The existence, at least in the extended sense, of a solution of (2.1) corresponding to ϕ at t_o and defined on some interval $[t_o,t_o + \tau_2)$, $\tau_2 > 0$, follows from Theorem 3.3. On the other hand, our hypothesis that $G(t,\psi(\alpha_t))$ in a right-continuous function of t implies that any solution in the extended sense is a solution in the usual sense. Hence we have only to establish uniqueness and continuous dependence on initial data.

Suppose x_1 and x_2 are two solutions of (2.1) corresponding to ϕ_1 and ϕ_2 in $U_{t_o}(B)$. Then we have

$$x_1(t) - x_2(t) = \phi_1(t_o) - \phi_2(t_o) + \int_{t_o}^{t} (G(\tau,x_1(\alpha_\tau)) - G(\tau,x_2(\alpha_\tau)))d\tau \tag{4.3}$$

for t in some interval $[t_o,t_o + \tau_1]$, $\tau_1 > 0$. Using condition (4.2) we have

$$| x_1(t) - x_2(t) | \leq | \phi_1(t_o) - \phi_2(t_o) | + \int_{t_o}^{t} |\mathcal{G}(\tau,x_1(\beta_\tau)) - \mathcal{G}(\tau,x_2(\beta_\tau))| d\tau$$

$$+ \int_{t_o}^{t} K(\tau) || x_1(\gamma_\tau) - x_2(\gamma_\tau) ||,$$

where $K(t)$ is a positive integrable function. Defining a function g by the formula

$$g(t) = |\phi_1(t_o) - \phi_2(t_o)| + ||\phi_1 - \phi_2||_{t_o} \int_{t_o}^{t} K(\tau)d\tau + \int_{t_o}^{t} | \circledast(\tau,x_1(\beta_\tau)) - \circledast(\tau,x_2(\beta_\tau))|d\tau$$

and using the fact that

$$||x_1(\gamma_t) - x_2(\gamma_t)|| \leq || \phi_1 - \phi_2||_{t_o} + ||x_1(\gamma_t \cap [t_o, t]) - x_2(\gamma_t \cap [t_o, t])|| \quad , \qquad (4.4)$$

we have that

$$|x_1(t)-x_2(t)| \leq g(t) + \int_{t_o}^{t} K(\tau)||x_1(\gamma_\tau \cap [t_o,\tau]) - x_2(\gamma_\tau \cap [t_o,\tau])|| \, d\tau . \qquad (4.5)$$

By our conditions on $||\ \ ||$ it follows that there exists a constant b such that

$$||x_1(\gamma_t \cap [t_o, t]) - x_2(\gamma_t \cap [t_o, t])|| \leq b(g(t) + \int_{t_o}^{t} K(\tau) \, ||x_1(\gamma_\tau \cap [t_o,\tau]) -$$

$$x_2(\gamma_\tau \cap [t_o, \tau])|| d\tau)$$

Now using Gronwall's lemma it follows that

$$||x_1(\gamma_t \cap [t_o,t])-x_2(\gamma_t \cap [t_o,t])|| \leq bg(t) + b^2 \int_{t_o}^{t} K(\tau)g(\tau) \exp \left(b\int_{\tau}^{t} K(s)ds\right)d\tau \ . \quad (4.6)$$

Choosing $t - t_o \leq \eta_1 = \min \{\tau_1,\eta\}$ and utilizing our hypothesis on ⊛ and Lebesgue's theorem on majorized sequences, we may conclude that for every $\varepsilon > 0$ we can choose $\delta > 0$ such that $||\phi_1-\phi_2|| < \delta$ implies $g(t) < \varepsilon$. Hence clearly (4.6) implies uniqueness and continuous dependence of solutions of (2.1) with respect to initial data for $t - t_o \leqq \eta_1$. If $\eta_1 < \tau_1$ we may replace equation (4.3) by the equation

$$x_1(t)-x_2(t) = x_1(t_o + \eta)-x_2(t_o + \eta) + \int_{t_o+\eta}^{t} (G(\tau,x_1(\alpha_\tau))-G(\tau,x_2(\alpha_\tau)))d\tau,$$

and repeating our argument, uniqueness and continuous dependence of solutions with respect to initial data may be extended for $t - t_o \leqq \min \{\tau_1,2\eta_1\}$.

If k is the smallest integer such that $k\eta_1 \leqq \tau_1$, then the indicated step by step procedure may be repeated k times to establish uniquesness and continuous dependence for $t - t_o \leqq \tau_1$ and complete our proof.

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